

Injective Labeled Oriented Trees are Aspherical

Jens Harlander and Stephan Rosebrock

December 11, 2012

Abstract

A labeled oriented tree (LOT) is called *injective*, if each generator occurs at most once as an edge label. We show that compressed injective labeled oriented trees are vertex aspherical. This implies that all injective labeled oriented trees are aspherical. In order to prove this result we introduce a new relative test on vertex asphericity based on a lemma of Stallings. A more general version of our main result shows vertex asphericity for a class of non-injective LOTs. Our methods yield π_1 -injectivity of sub-LOTs of injective labeled oriented trees and give the asphericity of long alternating virtual knots.

1 Introduction

This article is concerned with the Whitehead conjecture, which states that a subcomplex of an aspherical 2-complex is aspherical. See Bogley [1] and Rosebrock [7] for surveys. The conjecture originally arose in the context of knot theory. The Wirtinger presentation of a knot gives rise to a 2-complex that is a subcomplex of a contractible 2-complex. Thus, an affirmative answer to the conjecture implies the asphericity of knot complements in the 3-sphere. Labeled oriented trees are a way to record presentations that generalize Wirtinger presentations for knots and injective labeled oriented trees generalize Wirtinger presentations of alternating knots. In fact, labeled oriented trees play a central role in understanding the Whitehead conjecture. Howie [4] showed that the finite case of the Whitehead conjecture reduces, up to the Andrews-Curtis conjecture, to the statement that presentations arising from labeled oriented trees are aspherical.

A *labeled oriented graph* (LOG) is an oriented graph \mathcal{G} on vertices $\{1, \dots, k\}$, where each oriented edge is labeled by a vertex. Associated with it is the *LOG-presentation* $P = P(\mathcal{G})$ on generators x_1, \dots, x_k in one-to-one correspondence with the vertices and relators in one-to-one correspondence with edges. For an edge with initial vertex i , terminal vertex j , labeled l , we add a relation $x_i x_l = x_l x_j$. A *LOG-complex* $K(P)$ is the standard 2-complex associated with the LOG-presentation P , and a *LOG-group* $G(P)$, is the

group defined by the LOG-presentation. We say a labeled oriented graph is *aspherical* if its associated LOG-complex is aspherical. A labeled oriented graph is called *injective* if each vertex label occurs at most once as an edge label. A *labeled oriented tree* (LOT) is a labeled oriented graph where the underlying graph is a tree.

The following is the main result of this article.

Theorem 1.1 *Injective labeled oriented trees are aspherical.*

We will need the following additional terminology on labeled oriented graphs. A *sub-LOG* of a labeled oriented graph \mathcal{G} is a subgraph \mathcal{H} such that each edge label of \mathcal{H} is a vertex label of \mathcal{H} . A sub-LOG \mathcal{H} is *proper* if it contains at least one edge but is not all of \mathcal{G} . A labeled oriented graph is called *compressed* if every edge contains three different labels. It is called *boundary reducible* if there is a boundary vertex label that does not occur as edge label, and *boundary reduced* otherwise. A labeled oriented graph is called *interior reducible* if there is a vertex with two adjacent edges with the same label that either point away or towards that vertex. A labeled oriented graph which is boundary reduced, interior reduced and compressed is called *reduced*.

We end this introduction with an overview of how the article is structured. In Section 2 we review basic notions of combinatorial asphericity. Section 3 contains a new relative asphericity test based on a lemma of Stallings. This section forms the technical core of the paper and is also of interest outside the study of labeled oriented trees. Section 4 introduces the concept of altered LOT-presentations, contains Theorem 4.2 which is a result on orientations of LOTs and a graph-theoretic Lemma needed in Section 5. Theorem 4.2 was proven by Huck and Rosebrock [6] and is used to show that prime injective labeled oriented trees are aspherical (prime means that the labeled oriented tree does not contain sub-LOTs). Section 5 contains the proof of Theorem 2.1 which directly implies Theorem 1.1. In Section 6 we generalize Theorem 2.1 to a class of non-injective LOTs, show that the group of a sub-LOT of an injective labeled oriented tree \mathcal{P} is a subgroup of the LOT-group of \mathcal{P} and we close with an application to virtual knots.

2 Combinatorial notions of asphericity

Let K be a finite 2-complex. In this article K will always be the standard 2-complex of a finite presentation. So we assume in the following that K has a single vertex. A *spherical diagram* over K is a piecewise linear map $f: C \rightarrow K$, where C is a cell decomposition of the 2-sphere and f maps open cells of C homeomorphically to open cells of K . A cell of C will be

labeled by its image cell under f . The 1-cells also get their orientation from the 1-cells of K . In this way C itself carries all the information of the map $f: C \rightarrow K$ and we often speak of the “diagram C ”.

Let v be a vertex in a spherical diagram $f: C \rightarrow K$. This induces a map $\bar{f}: Lk(C, v) \rightarrow Lk(K)$, where $Lk(C, v)$ is the link of v in C and $Lk(K)$ is the link of the unique vertex in K . Note that $Lk(C, v)$ is a circle and the image of that circle is a cycle $\alpha(v)$, a closed edge path in $Lk(K)$. We refer to the edges in $Lk(K)$ as *corners* since they arise from the corners in the 2-cells of K . Suppose that $\alpha(v) = \alpha_1 \dots \alpha_q$. We say that the diagram C is reduced at v if the cycle $\alpha(v)$ is freely reduced, that is $\alpha_{i+1} \neq \bar{\alpha}_i$, $i = 1, \dots, q$ ($i \bmod q$), where $\bar{\alpha}_i$ is the corner α_i with opposite orientation. The diagram C is *reduced* if it is reduced at every vertex. A 2-complex is called *diagrammatically reducible*, DR, if it does not admit reduced spherical diagrams. Note that the diagram C is reduced if and only if it does not contain a pair of 2-cells that is mapped to the same 2-cell of K by folding the pair across a shared edge.

In the following we will also need a weakening of the DR concept. We say that the diagram C is *homology reduced* at v if the cycle $\alpha(v)$ is *homology reduced*, that is $\alpha(v)$ does not contain the same corner α in opposite orientation. We say that C is *vertex reduced* if it is homology reduced at all its vertices. Following [5] a 2-complex K is called *vertex aspherical*, VA, if it does not admit a vertex reduced spherical diagram over K . Note that the diagram C is vertex reduced if and only if it does not contain a pair of 2-cells that is mapped to the same 2-cell of K by folding the pair across a shared vertex.

A labeled oriented graph is called DR (VA) if the corresponding standard 2-complex is DR (VA), respectively. A 2-complex that is VA is aspherical (see [5]).

Theorem 2.1 *Compressed injective labeled oriented trees are VA.*

Our main result, Theorem 1.1, follows from the above theorem and the fact that every labeled oriented tree can be transformed into a reduced labeled oriented tree, and the homotopy type of the associated 2-complex remains unchanged under this transformation.

3 A relative test for vertex asphericity

The following result is due to Stallings [8]:

Lemma 3.1 *Each cell decomposition of the 2-sphere with oriented 1-cells contains at least two consistent items.*

Here, a consistent item is either a *sink* (a vertex with all adjacent edges pointing toward it), or a *source* (a vertex with all adjacent edges pointing away from it) or a *consistently oriented 2-cell* (a 2-cell whose boundary consists of edges, all oriented clockwise or all oriented counterclockwise).

Let $P = \langle x_1, \dots, x_k \mid r_1, \dots, r_m \rangle$ be any finite presentation. The *Whitehead graph* $W(P)$ is the link $Lk(K(P))$ of the only vertex of the standard 2-complex $K(P)$ constructed from the presentation. It is a nonoriented graph consisting of a pair of vertices x_i^+ and x_i^- for each generator x_i of P , which correspond to the beginning and the end of the oriented loop labeled x_i in $K(P)$. The edges of $W(P)$, also called *corners*, are the intersections of the polygonal 2-cells with the boundary of a regular neighborhood of the vertex of $K(P)$. The *positive graph* $W^+(P) \subset W(P)$ is the full subgraph on the vertices x_1^+, \dots, x_k^+ , the *negative graph* $W^-(P) \subset W(P)$ is the full subgraph on the vertices x_1^-, \dots, x_k^- .

Let $P = \langle x_1, \dots, x_k \mid r_1, \dots, r_m \rangle$ be a finite presentation. If all defining relations contain generators with positive and negative exponents and the positive, and the negative graph of $W(P)$ are trees, then $K(P)$ is DR. Indeed, a reduced spherical diagram over $K(P)$ can not contain a sink, a source, or a consistently oriented 2-cell in contradiction to Lemma 3.1. This application of Stallings Lemma is well known.

In this article we will use Stallings result in a relative setting. Let $P = \langle x_1, \dots, x_k \mid r_1, \dots, r_m \rangle$ be a finite presentation such that each relator r_i has exponent sum 0. Note that this is equivalent to the requirement the map $x_i \mapsto 1, i = 1, \dots, k$, defines an epimorphism from $G(P)$, the group defined by the presentation, to the infinite cyclic group \mathbb{Z} . Let $T = \{T_1, \dots, T_n\}$ be a set of disjoint sub-presentations of P . So each T_i is a sub-presentation and the generating sets of T_i and T_j are disjoint subsets of $\{x_1, \dots, x_k\}$ in case $i \neq j$. The complex $K(T)$ is a sub-complex of $K(P)$. Let U_i be the set of words of exponent sum 0 in the generators and their inverses of T_i . Note that we do not assume that the words in U_i are freely reduced. Let $P/T = \langle x_1, \dots, x_k \mid r_1, \dots, r_m, U_1, \dots, U_n \rangle$. The presentation P/T is infinite and in the group $G(P/T)$ the generators of each T_i are identified. We refer to the words in $U = U_1 \cup \dots \cup U_n$ as T^* -relations. Note that the relations of the sub-presentations T_i are contained in U_i and hence are also T^* -relations. Let $t_{i,1}, \dots, t_{i,l_i}$ be the generators of some T_i . Then the subgraph of the Whitehead graph $W(P/T)$ spanned by the vertices $t_{i,1}^\pm, \dots, t_{i,l_i}^\pm$ contains the complete graph on these vertices. In fact, every pair of vertices is connected by infinitely many edges, and at every vertex there are attached infinitely

many loops. We call the presentation P/T a *relative* presentation.

A cycle $\alpha = \alpha_1 \dots \alpha_q$ in the Whitehead graph $W(P/T)$, each α_i being a corner of $W(P/T)$, is called *admissible* if

1. At least one corner α_i comes from a relation which is not a T^* -relation,
2. if α_i is a corner of a T^* -relation then α_{i-1} and α_{i+1} ($i \bmod q$) come from relators which are not T^* -relations.

Definition 3.2 Let P be a finite presentation, where each defining relation has exponent sum 0. Let $T = \{T_1, \dots, T_n\}$ be a set of disjoint sub-presentations. The relative presentation P/T is said to satisfy the relative Stallings-test, if there is no admissible homology reduced cycle in the positive graph or in the negative graph of $W(P/T)$.

Lemma 3.3 If a relative presentation P/T satisfies the relative Stallings-test, then there does not exist a vertex reduced spherical diagram $f: C \rightarrow K(P/T)$ where all vertex cycles $\alpha(v)$ are admissible.

Proof: Assume there is a vertex reduced spherical diagram where all vertex cycles are admissible. Since P/T satisfies the relative Stallings-test there are no admissible homology reduced cycles in the positive graph or in the negative graph of $W(P/T)$. So the diagram can not have a source or a sink. Since we assume that each defining relator of P has exponent sum 0, the relators of P/T will have exponent sum 0 also. So the diagram can not have a consistently oriented region. This contradicts Lemma 3.1. \square

Theorem 3.4 If P/T satisfies the relative Stallings-test and $K(T)$ is VA, then $K(P)$ is VA.

Proof: Assume there exists a vertex reduced spherical diagram $f: C \rightarrow K(P)$. Note that C can not be a diagram over $K(T)$, that is $f(C)$ can not be contained in $K(T)$, because we assume $K(T)$ to be VA. Observe also that two 2-cells of C which map to two different $K(T_i)$ never have a 1-cell in common because different T_i do not share generators.

Suppose that C contains a vertex v where the vertex cycle $\alpha(v) = \alpha_1 \dots \alpha_q$ is not admissible. If $\alpha_i \alpha_{i+1} \dots \alpha_{i+k}$ is a sub-path of $\alpha(v)$ consisting of corners in relators of the sub-presentation T , then we can replace the gallery of 2-cells in C that contain $\alpha_i \alpha_{i+1} \dots \alpha_{i+k}$ by a single 2-cell whose boundary word is a T^* -relation. Note that this process reduces the number of 2-cells in C , but does not affect 2-cells with boundary words that are not T^* -relations.

The process has to terminate. In the end we obtain a spherical diagram $f_T: C_T \rightarrow K(P/T)$ where each vertex cycle $\alpha(v)$ is admissible. Suppose C_T contains a vertex v where the cycle $\alpha(v) = \alpha_1 \dots \alpha_q$ is not homology reduced. So $\alpha_i = \bar{\alpha}_j$ for some i, j . Note that α_i can not be a corner of a cell for a relation of $P - T$ because that would imply that the original diagram C is not vertex reduced. Thus α_i has to be a corner of a cell for a T^* -relation. We can remove the two cells that contain the corners α_i and α_j and replace it with a single cell whose boundary word is a (freely reducible) T^* -relation (see Figure 1).

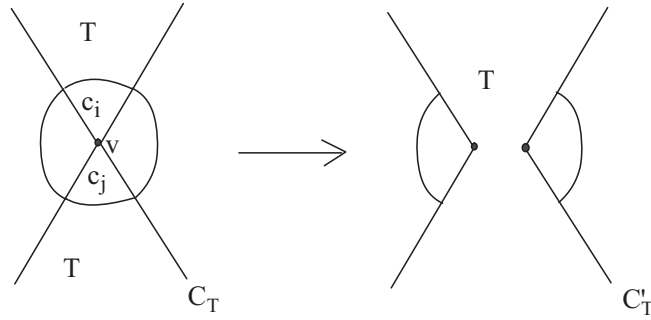


Figure 1: Splitting a cycle of C_T

In this fashion we built a spherical diagram $f'_T: C'_T \rightarrow K(P/T)$ which is vertex reduced and all vertex cycles are admissible. But such a diagram does not exist by Lemma 3.3. Thus our assumption that there is a vertex reduced spherical diagram $f: C \rightarrow K(P)$ is false and hence $K(P)$ is VA. \square

Theorem 3.5 *Let P be a finite presentation and let $T = \{T_1, \dots, T_n\}$ be a set of disjoint sub-presentations. Suppose P/T satisfies the relative Stallings-test. Then the map $\pi_1(K(T_i)) \rightarrow \pi_1(K(P))$, induced by inclusion, is injective for every $i = 1, \dots, n$.*

Proof: Suppose the map $\pi_1(K(T_i)) \rightarrow \pi_1(K(P))$ is not injective. Then there exists a word u in the generators of T_i , $u \neq 1$ in $\pi_1(K(T_i))$, that is the boundary of a vertex reduced disc diagram $g: D \rightarrow K(P)$. Note that D has to contain 2-cells that are not mapped to $K(T)$ because the map $\pi_1(T_i) \rightarrow \pi_1(T) = \pi_1(T_1) * \dots * \pi_1(T_n)$ is injective. Since the word u itself is a T^* -relation we can attach a disc D' to D and obtain a spherical diagram $f: C \rightarrow K(P/T)$ that is vertex reduced. We now apply the same process as used in the proof of Theorem 3.4 to obtain a spherical diagram $f': C' \rightarrow K(P/T)$ that is vertex reduced and all vertex cycles $\alpha(v)$ are admissible.

This contradicts Lemma 3.3. \square

4 Altered LOT-presentations and orientations

Let P be a finite presentation and T be a finite (possibly empty) set of disjoint sub-presentations of P . Let S be a subset of the set of generators of P . In every relation of the relative presentation P/T replace x_i by x_i^{-1} if $x_i \in S$. This results in a new presentation $(P/T)_S$. Note that this change in orientation results in a homeomorphism $\phi: K(P/T) \rightarrow K((P/T)_S)$ on the corresponding standard 2-complexes.

Lemma 4.1 *$K(P/T)$ is VA if and only if $K((P/T)_S)$ is VA. In particular, if T is empty, $K(P)$ is VA if and only if $K(P_S)$ is VA.*

Proof: There is a bijection between spherical diagrams over $K(P/T)$ and spherical diagrams over $K((P/T)_S)$ that preserves the notion of reducibility. The bijection simply reverses the orientation of an edge labeled x_i in the diagram, if $x_i \in S$. \square

Let \mathcal{P} be a labeled oriented tree and \mathcal{T} be a (possibly empty) set of labeled oriented subtrees. Let P and T be the corresponding LOT-presentations. Let S be a subset of the set of those generators in P that occur as edge labels in $\mathcal{P} - \mathcal{T}$. We call the presentation $(P/T)_S$ an *altered relative LOT-presentation*. Relators of an altered relative LOT-presentation are either LOT-relators, or are of the form $x_i^{-1}x_p = x_px_q$, $x_i \in S$, which we call an *altered LOT-relator*, or are altered and unaltered T^* -relators.

We say a labeled oriented graph \mathcal{Q} is a *reorientation* of a labeled oriented graph \mathcal{P} if \mathcal{Q} is obtained from \mathcal{P} by changing the orientation of a subset of edges of \mathcal{P} .

The next result was implicitly proved in [6]. See the beginning of Section 3 of that paper.

Theorem 4.2 *Let \mathcal{P} be a labeled oriented tree that is compressed and injective and does not contain a boundary reducible sub-LOT. Then there is a reorientation \mathcal{Q} of \mathcal{P} such that the positive and the negative part of the Whitehead graph, $W^+(Q)$ and $W^-(Q)$, are trees, where Q is the LOT-presentation associated with \mathcal{Q} .*

We end this section with a graph theoretic lemma that will be needed in the next section, where we give the proof of the main theorem.

Lemma 4.3 *Let Γ be a graph and $\Gamma_1, \dots, \Gamma_n$ be a set of disjoint connected subgraphs. Let Γ' be the graph obtained by collapsing each subgraph Γ_i to a vertex $g_i \in \Gamma_i$. If Γ' is a tree, then a homology reduced cycle in Γ is contained in one of the Γ_i .*

Proof: Let X be the union of the Γ_i and Y be the smallest subgraph of Γ that contains $\Gamma - X$. The intersection $X \cap Y$ is a set of vertices. We first show that $H_0(X \cap Y) \rightarrow H_0(X) \oplus H_0(Y)$ is injective. In order to see this consider two vertices a and b in $X \cap Y$. Assume first that they belong to the same Γ_i . If there were a path connecting a and b in Y then this path would produce a cycle in Γ' , which is impossible since Γ' is a tree. Thus a and b determine different elements in $H_0(Y)$. If $a \in \Gamma_i$ and $b \in \Gamma_j$, $i \neq j$, then a and b determine different elements in $H_0(X)$, because we assumed the Γ_i to be disjoint. Consider the Mayer-Vietoris sequence

$$0 \rightarrow H_1(X) \oplus H_1(Y) \rightarrow H_1(\Gamma) \rightarrow H_0(X \cap Y) \rightarrow H_0(X) \oplus H_0(Y).$$

The rightmost map is injective, hence we have an isomorphism

$$H_1(X) \oplus H_1(Y) \rightarrow H_1(\Gamma).$$

Now $H_1(X) = \bigoplus_{i=1}^n H_1(\Gamma_i)$ and $H_1(Y) = 0$, because Γ' is a tree and hence Y is a forest. So the inclusion induced map $\bigoplus_{i=1}^n H_1(\Gamma_i) \rightarrow H_1(\Gamma)$ is an isomorphism. Thus every element Z of $H_1(\Gamma)$ is uniquely represented by a sum $Z = Z_1 + \dots + Z_n$, where each Z_i is a sum of homology reduced cycles in Γ_i . In particular, a homology reduced cycle in Γ has to be contained in one of the Γ_i . \square

5 Proof of Theorem 2.1

Let \mathcal{P} be a compressed injective labeled oriented tree. In this section we will show that \mathcal{P} is VA. We will show this by induction on the number of vertices. If \mathcal{P} consists of a single vertex the result is true. Let $\{\mathcal{T}_1, \dots, \mathcal{T}_n\}$ be the (possibly empty) set of maximal proper sub-LOTs of \mathcal{P} . Note that every \mathcal{T}_i is compressed and injective and contains fewer vertices than \mathcal{P} . Hence, by induction, each \mathcal{T}_i is VA.

Case 1. Suppose that for some i, j we have $\mathcal{T}_i \cap \mathcal{T}_j \neq \emptyset$.

Note that then $\mathcal{T}_i \cup \mathcal{T}_j = \mathcal{P}$, because otherwise this union would be a proper sub-LOT, contradicting maximality of the \mathcal{T}_k . Thus we are in the situation that $\mathcal{T}_1 \cup \mathcal{T}_2 = \mathcal{P}$. We will show that there exist sub-LOTs \mathcal{T}'_1 and \mathcal{T}'_2 so that the intersection $\mathcal{T}'_1 \cap \mathcal{T}'_2$ is a single vertex and the union $\mathcal{T}'_1 \cup \mathcal{T}'_2 = \mathcal{P}$. The

intersection $\mathcal{T}_1 \cap \mathcal{T}_2$ is a sub-LOT. Indeed, if b is an edge label in $\mathcal{T}_1 \cap \mathcal{T}_2$, then b has to be a vertex of \mathcal{T}_1 , because \mathcal{T}_1 is a sub-LOT, and b has to be a vertex of \mathcal{T}_2 , because \mathcal{T}_2 is a sub-LOT. Thus b is a vertex of $\mathcal{T}_1 \cap \mathcal{T}_2$. Suppose that neither $\mathcal{T}_1 - (\mathcal{T}_1 \cap \mathcal{T}_2)$ nor $\mathcal{T}_2 - (\mathcal{T}_1 \cap \mathcal{T}_2)$ is a sub-LOT. Then $\mathcal{T}_i - (\mathcal{T}_1 \cap \mathcal{T}_2)$ contains an edge label b_i that is not a vertex in $\mathcal{T}_i - (\mathcal{T}_1 \cap \mathcal{T}_2)$ and hence is a vertex in $\mathcal{T}_1 \cap \mathcal{T}_2$, $i = 1, 2$. But that implies that $\mathcal{T}_1 \cap \mathcal{T}_2$ contains two vertices that do not occur as edge labels, which is not possible in the injective setting. Thus, we may assume that $\mathcal{T}_1 - (\mathcal{T}_1 \cap \mathcal{T}_2)$ is a sub-LOT. Set $\mathcal{T}'_1 = \mathcal{T}_1 - (\mathcal{T}_1 \cap \mathcal{T}_2)$ and $\mathcal{T}'_2 = \mathcal{T}_2$.

So in case 1 we may assume that $\mathcal{P} = \mathcal{T}_1 \cup \mathcal{T}_2$ and the intersection $\mathcal{T}_1 \cap \mathcal{T}_2 = \{x\}$ is a single vertex x . In this case our main result follows from the next lemma.

Lemma 5.1 *Suppose that \mathcal{P} is a compressed injective labeled oriented tree. Assume $\mathcal{P} = \mathcal{T}_1 \cup \mathcal{T}_2$ and $\mathcal{T}_1 \cap \mathcal{T}_2 = \{x\}$. If both \mathcal{T}_i are VA, then so is \mathcal{P} .*

Proof: Assume there exists a vertex reduced spherical diagram $f : C \rightarrow K(P)$. This diagram can not map entirely into $K(\mathcal{T}_1)$ or $K(\mathcal{T}_2)$ (VA assumption). Let $D \subset C$ be a maximal region, homeomorphic to a disc, that maps entirely into $K(\mathcal{T}_1)$ or $K(\mathcal{T}_2)$. Without loss of generality we assume that D maps into $K(\mathcal{T}_1)$. The boundary word of D is a word in $x^{\pm 1}$. Since x generates an infinite cyclic group, there has to occur a free reduction $x^\epsilon x^{-\epsilon}$ at a vertex v in the boundary of D . Consider the case when \mathcal{T}_1 does not contain x as an edge label. Since \mathcal{T}_1 contains only one 2-cell with boundary edge x , the region D contains that 2-cell in opposite orientations at the vertex v . But that implies that C is not vertex reduced. Next assume that \mathcal{T}_1 does contain an edge labeled x . \mathcal{P} is injective so \mathcal{T}_2 contains exactly one 2-cell with x in its boundary. Then consider the 2-cells in $C - D$ that contain the edge reduction $x^\epsilon x^{-\epsilon}$. These are 2-cells of \mathcal{T}_2 , and hence are the same 2-cell with opposite orientations. This implies that C is not vertex reduced, contradicting our assumption. \square

Case 2. The \mathcal{T}_i , $i = 1, \dots, n$, are pairwise disjoint.

Let \mathcal{P}' be the labeled oriented tree obtained by collapsing each \mathcal{T}_i to the vertex $t_i \in \mathcal{T}_i$ that is not an edge label in \mathcal{T}_i . Note that \mathcal{P}' is compressed since \mathcal{P} is compressed and the sub-LOTs \mathcal{T}_i are assumed to be maximal. Furthermore \mathcal{P}' is injective because we assumed \mathcal{P} to be injective. So Theorem 4.2 implies that there is a reorientation \mathcal{Q}' of \mathcal{P}' , such that the positive and the negative Whitehead graph of $K(\mathcal{Q}')$ are trees. Let \mathcal{Q} be a reorientation of \mathcal{P} where edge orientations coincide with \mathcal{P} on the edges of $\mathcal{T} = \mathcal{T}_1 \cup \dots \cup \mathcal{T}_n$, and edge orientations coincide with \mathcal{Q}' on the edges of $\mathcal{P} - \mathcal{T}$. Let P, Q, Q' ,

and T be LOT-presentations associated with \mathcal{P} , \mathcal{Q} , \mathcal{Q}' , and \mathcal{T} , respectively.

Lemma 5.2 *The relative presentation Q/T satisfies the relative Stallings-test.*

Proof: Relators in Q have exponent sum zero and therefore relators in Q/T also. It remains to show that there are no admissible homology reduced cycles in $W^+(Q/T)$ or $W^-(Q/T)$. Note that $W^+(Q')$ is obtained from $W^+(Q/T)$ by collapsing each subgraph $W^+(T_i/T_i)$ to t_i^+ , for each $i = 1, \dots, n$. Thus, since $W^+(Q')$ is a tree, we are in the situation of Lemma 4.3. It follows that a homology reduced cycle in $W^+(Q/T)$ is contained in some $W^+(T_i/T_i)$. Since no $W^+(T_i/T_i)$ contains admissible cycles, we see that $W^+(Q/T)$ does not contain such cycles either. The argument for $W^-(Q/T)$ is analogous. \square

We want to show that $K(P)$ is VA. Let S be the set of edge labels on those edges that change orientation by passing from \mathcal{P} to \mathcal{Q} . Let $(P/T)_S$ be the altered relative LOT-presentation (see Section 4 for definitions). Suppose $f: C \rightarrow K(P)$ is a vertex reduced spherical diagram. It does not entirely map into $K(T)$ because $K(T)$ is VA by induction. As in the proof of Theorem 3.4 we can make the diagram into one $f': C' \rightarrow K(P/T)$ that is vertex reduced and all vertex cycles are admissible. Now apply the homeomorphism $K(P/T) \rightarrow K((P/T)_S)$ to produce a vertex reduced diagram over $K((P/T)_S)$ with admissible vertex cycles. If we could show that $(P/T)_S$ satisfies the relative Stallings test, such a diagram does not exist and we would have arrived at a contradiction to the assumption that there is a vertex reduced spherical diagram $f: C \rightarrow K(P)$. The problem is that $(P/T)_S$ can contain relations with only positive (or only negative) exponents. The above line of reasoning will have to be refined. We will show however that $W^+((P/T)_S)$ and $W^-((P/T)_S)$ do not contain admissible homology reduced cycles. So $(P/T)_S$ does indeed satisfy the first part of the relative Stallings test.

Lemma 5.3 *The Whitehead graphs $W((P/T)_S)$ and $W(Q/T)$ are equal. Also, the Whitehead graphs $W(P'_S)$ and $W(Q')$ are equal.*

Proof: Notice first that the graph $W(T/T) \subset W(Q/T)$ contains the complete graph on its vertices. In fact, every pair of vertices is connected by infinitely many edges, and at every vertex there are attached infinitely many loops. Exactly the same is true for the subgraph $W((T/T)_S) \subseteq W((P/T)_S)$.

If $x_i x_p = x_p x_q$ is a relation in $Q - T \subset Q/T$ and neither x_i nor x_q are in S , then exactly the same relation occurs in $(P - T)_S \subset (P/T)_S$.

Consider a relation $x_i x_p = x_p x_q$ in $Q - T \subset Q/T$, where $x_i \in S$, but $x_q \notin S$. The corresponding relation in $(P - T)_S \subset (P/T)_S$ is of the form $x_i^{-1} x_p = x_p x_q$. Note that the corners these relations contribute to the Whitehead graphs are exactly the same.

Consider a relation $x_i x_p = x_p x_q$ in $Q - T \subset Q/T$, where $x_i \in S$ and $x_q \in S$. The corresponding relation in $(P - T)_S \subset (P/T)_S$ is of the form $x_i^{-1} x_p = x_p x_q^{-1}$. Again note that the corners these relations contribute to the Whitehead graphs are exactly the same.

Thus we have shown that $W((P/T)_S) = W(Q/T)$. The statement $W(P'_S) = W(Q')$ can be shown by the same arguments. In fact, this is a special case of the above where T is empty. \square

Since Q/T satisfies the relative Stallings-test $(P/T)_S$ satisfies the first part of the relative Stallings-test by the lemma just shown. That is, $W^+((P/T)_S)$ and $W^-((P/T)_S)$ do not contain admissible homology reduced cycles. So a vertex reduced spherical diagram with admissible vertex cycles $f: C \rightarrow K((P/T)_S)$ can not contain a sink or a source.

Lemma 5.4 *If $f: C \rightarrow K((P/T)_S)$ is a vertex reduced spherical diagram then $f(C)$ is contained in $K((T/T)_S)$.*

Proof: Suppose $f: C \rightarrow K((P/T)_S)$ is a vertex reduced spherical diagram, where $f(C)$ is not contained in $K((T/T)_S)$. We can apply the same process as used in the proof of Theorem 3.4 to produce a vertex reduced spherical diagram $f_T: C_T \rightarrow K((P/T)_S)$ where each vertex cycle $\alpha(v)$ is admissible. Thus C_T can not contain a sink or a source vertex. It may contain consistently oriented regions. For instance if a is a generator in T_i and $a \in S$ then a relator $a^{-1}b$ in T_i , $b \notin S$, leads to a 2-cell e with boundary word ab in C_T . See Figure 2. Note that e is consistently oriented.

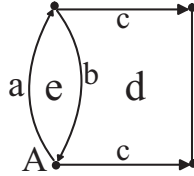


Figure 2: Two regions of C_T

Consistently oriented cells $e \in C_T$ come from relations in $(T/T)_S$ that are positive or negative words. Suppose w is such a word. Then w is a word in the generators of some T_i and it must contain a generator $a \in S$ since all

relators in T/T have exponent sum zero. Note that since we assumed the labeled oriented tree \mathcal{P}' to be injective, the subtree \mathcal{T}_i can contain at most one vertex that occurs as an edge label in $\mathcal{P} - \mathcal{T}$. So T_i can contain at most one generator in S . Thus any generator $b \neq a$ that occurs in w is not an edge label in $\mathcal{P} - \mathcal{T}$. It follows that every edge label $b \neq a$ in the boundary of the region e is not an edge label in $\mathcal{P} - \mathcal{T}$. Let d be the 2-cell of $P - T$ adjacent to such an edge with label b . See Figure 2. Note that b occurs exactly once in the boundary of d , otherwise b would have to be an edge label in $\mathcal{P} - \mathcal{T}$. It follows that d contains oppositely oriented edges labeled by $c \neq b$. We now erase the edge labeled b from the union $e \cup d$, creating a region whose boundary is not consistently oriented.

We have to make sure that this process of erasing edges does not produce a sink or a source in the diagram. Assume a source would be created at the vertex A of Figure 2 by erasing the edge b . The result is shown in Figure 3.

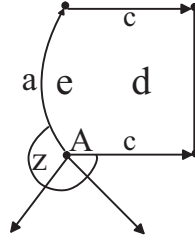


Figure 3: A source creation

In $W(P'_S)^+$ the vertices a^+, b^+ are identified to a vertex t_i^+ . The 2-cell d gives rise to a corner s from c^+ to b^+ in $W(P_S)$ and therefor from c^+ to t_i^+ in $W(P'_S)$. If we have a source at A then there must be a path z from c^+ to a^+ in $W^+(P_S)$ and therefor a path from c^+ to t_i^+ in $W^+(P'_S)$. If $s \neq z$ this leads to a reduced cycle in $W^+(P'_S)$. Since the Whitehead graphs $W(P'_S)$ and $W(Q')$ are equal (Lemma 5.3) and $W^+(Q')$ is a tree we have a contradiction.

In case $s = z$ we are in a situation as depicted in Figure 4. Erasing the edge labeled b between e and d would lead to a source at the vertex A . In this case we do not erase the b -edge but we move along the boundary of e until we reach a boundary edge not labeled a but followed by an edge labeled a . We erase that edge and create a region $e \cup d'$.

The c -edge in Figure 3 could also point the other direction, towards the T_i -region. See Figure 5. Erasing the edge labeled by b creates a sink at the vertex B . We can argue as above, using the fact that the negative graph of $W(P'_S)$ is a tree.

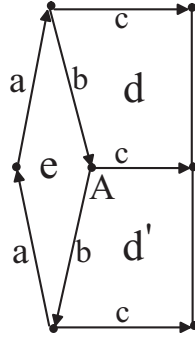


Figure 4: Exceptional case

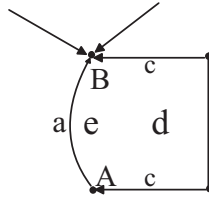


Figure 5: A sink creation

Repeating this edge removal process for all consistently oriented regions leads to a cell decomposition of the 2-sphere without consistent items. This is a contradiction to Lemma 3.1. This completes the proof of Lemma 5.4. \square

We can now complete the proof of Theorem 2.1. Suppose $f: C \rightarrow K(P)$ is a vertex reduced spherical diagram. Note that $f(C)$ can not be contained in $K(T)$, because $K(T)$ is VA. We can apply the homeomorphism $K(P) \rightarrow K(P_S)$ to obtain a vertex reduced spherical diagram $f': C' \rightarrow K(P_S)$ where $f'(C')$ is not contained in $K(T_S)$. Since $K(P_S)$ is a subcomplex of $K((P/T)_S)$, we have a vertex reduced spherical diagram $f': C' \rightarrow K((P/T)_S)$, where $f'(C')$ is not contained in $K((T/T)_S)$. But this contradicts Lemma 5.4.

6 Further consequences and applications

Observe that Theorem 2.1 is true for all compressed injective labeled oriented forests because each compressed injective labeled oriented forest is

a subcomplex of a compressed injective labeled oriented tree and subcomplexes inherit the property VA.

Case 2 in the proof given in Section 2.1 never uses the fact that the sub-LOTs \mathcal{T}_i are injective. We only use that the \mathcal{T}_i are VA. Hence the following result holds.

Theorem 6.1 *Let \mathcal{P} be a compressed labeled oriented tree and let $\mathcal{T} = \{\mathcal{T}_1, \dots, \mathcal{T}_n\}$ be the set of maximal proper sub-LOTs. Suppose that $\mathcal{P} \neq \mathcal{T}_1 \cup \mathcal{T}_2$. Let \mathcal{P}' be the labeled oriented tree where each \mathcal{T}_i is identified to a vertex t_i . If each \mathcal{T}_i is VA and \mathcal{P}' is injective then \mathcal{P} is VA.*

This theorem applies to many non-injective labeled oriented trees because a generator a might be an edge label in a sub-LOT \mathcal{T}_i and also in $\mathcal{P} - \mathcal{T}$. The \mathcal{T}_i may be VA for a variety of reasons without being injective. For instance they could satisfy small-cancellation conditions.

Theorem 6.2 *Suppose \mathcal{P} is an injective labeled oriented tree and \mathcal{H} is a sub-LOT. Then the LOT-group H of \mathcal{H} is a subgroup of the LOT-group G of \mathcal{P} .*

Proof: We prove the result by induction on the number of vertices of \mathcal{P} . If \mathcal{P} has only one vertex, then there is nothing to show. Suppose that the result holds for all injective labeled oriented trees with less vertices than \mathcal{P} . Let $\{\mathcal{T}_1, \dots, \mathcal{T}_n\}$ be the set of maximal proper sub-LOTs of \mathcal{P} . Let P , H , and T_i be the LOT-presentations associated with \mathcal{P} , \mathcal{H} , and \mathcal{T}_i , respectively. Note that it suffices to show that $\pi_1(K(T_i)) \rightarrow \pi_1(K(P))$ is injective for every $i = 1, \dots, n$. This is because \mathcal{H} is contained in some \mathcal{T}_i and we know that $\pi_1(K(H)) \rightarrow \pi_1(K(T_i))$ is injective by induction hypothesis, since \mathcal{T}_i has fewer vertices than \mathcal{P} .

Case 1. $\mathcal{P} = \mathcal{T}_1 \cup \mathcal{T}_2$. Note that $\mathcal{T}_1 \cap \mathcal{T}_2$ is a sub-LOT of both \mathcal{T}_1 and \mathcal{T}_2 and $\pi_1(K(\mathcal{T}_1 \cap \mathcal{T}_2)) \rightarrow \pi_1(K(T_i))$, $i = 1, 2$, is injective by induction hypothesis. Thus $\pi_1(K(P)) = \pi_1(K(T_1)) *_{\pi_1(K(\mathcal{T}_1 \cap \mathcal{T}_2))} \pi_1(K(T_2))$ is an amalgamated product. Hence $\pi_1(K(T_i)) \rightarrow \pi_1(K(P))$ is injective.

Case 2. The \mathcal{T}_i are pairwise disjoint. Let $T = T_1 \cup \dots \cup T_n$. As in the proof of Theorem 2.1 we pass to altered LOT-presentation P_S so that both $W^+((P/T)_S)$ and $W^-((P/T)_S)$ do not contain admissible homology reduced cycles. Note that it suffices to show that the inclusion induced map $\pi_1(K((T_i)_S)) \rightarrow \pi_1(K(P_S))$ is injective, because the homeomorphism $K(P_S) \rightarrow K(P)$ restricts to a homeomorphism $K((T_i)_S) \rightarrow K(T_i)$. Suppose u is a word in the generators of T_i which presents a non-trivial element in $\pi_1(K((T_i)_S))$ but does present the trivial element in $\pi_1(K(P_S))$. Then there

exists a vertex reduced disc diagram $g: D \rightarrow K(P_S)$ with boundary word u so that $g(D)$ is not contained in $K(T_S)$. Since u is a T_S^* -relation, we can attach a disc D' to D and obtain a vertex reduced spherical diagram $f: C \rightarrow K((P/T)_S)$ where $f(C)$ is not contained in $K((T/T)_S)$. This is a contradiction to Lemma 5.4. \square

We conclude this article with an application to long virtual knots. See Kauffman [3] for an overview of virtual knot theory. A virtual link diagram is a planar 4-regular graph with under- and over crossing information at some nodes. A virtual knot diagram is a virtual link diagram with only one link component. A long virtual knot diagram k is obtained by cutting a virtual knot diagram at a point on an edge, thus producing a graph that has exactly two nodes of valency one. These are the ends of the long virtual knot diagram k . A Wirtinger presentation $P(k)$ can be read off a long virtual knot diagram in the usual way. It is easy to see that $P(k) = P(\mathcal{P})$, where \mathcal{P} is a labeled oriented interval. More details on the connection between labeled oriented intervals and long virtual knots can be found in Harlander and Rosebrock [2]. We say a long virtual knot diagram is *aspherical*, if the standard 2-complex associated with the Wirtinger presentation is aspherical. A virtual knot diagram is *alternating* if one encounters over- and under-crossings in an alternating fashion when traveling along the diagram. A long alternating virtual knot diagram is obtained when cutting an alternating virtual knot diagram.

Corollary 6.3 *A long alternating virtual knot diagram k is aspherical.*

Proof: The labeled oriented interval that records the Wirtinger presentation of k is injective. The result follows from Theorem 2.1. \square

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Jens Harlander
Boise State University, USA
jensharlander@boisestate.edu

Stephan Rosebrock
Pädagogische Hochschule Karlsruhe, Germany
rosebrock@ph-karlsruhe.de